Computational Principles for High-dim Data Analysis

(Lecture Five)

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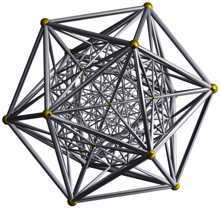 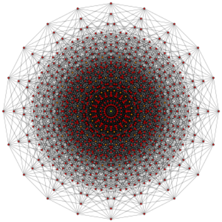 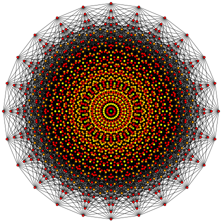

Convex Methods for Sparse Signal Recovery
(Towards Stronger Correctness Results)

1. Restricted Isometry Property (RIP)
2. Restricted Strong Convexity (RSC)
3. Success of ℓ^1 Minimization under RIP

"Algebra is but written geometry; geometry is but drawn algebra."
– Sophie Germain

From Incoherence to Isometry

Consider two columns $\boldsymbol{A}_{\mathsf{I}} = [\boldsymbol{a}_i \mid \boldsymbol{a}_j] \in \mathbb{R}^{m \times 2}$ of $\boldsymbol{A}$,

$$\boldsymbol{A}_{\mathsf{I}}^* \boldsymbol{A}_{\mathsf{I}} = \begin{bmatrix} 1 & \boldsymbol{a}_i^* \boldsymbol{a}_j \\ \boldsymbol{a}_j^* \boldsymbol{a}_i & 1 \end{bmatrix} \quad \mathbb{R}^{2 \times 2}. \tag{1}$$

If $|\boldsymbol{a}_i^* \boldsymbol{a}_j| \le \mu(\boldsymbol{A})$ is small, this matrix is well conditioned:

$$1 - \mu(\boldsymbol{A}) \;\le\; \sigma_{\min}(\boldsymbol{A}_{\mathsf{I}}^* \boldsymbol{A}_{\mathsf{I}}) \;\le\; \sigma_{\max}(\boldsymbol{A}_{\mathsf{I}}^* \boldsymbol{A}_{\mathsf{I}}) \;\le\; 1 + \mu(\boldsymbol{A}). \tag{2}$$

$\forall\, \mathsf{I}$ of size $\le k$,

$$1 - \underbrace{k\mu(\boldsymbol{A})}_{\delta} \;\le\; \sigma_{\min}(\boldsymbol{A}_{\mathsf{I}}^* \boldsymbol{A}_{\mathsf{I}}) \;\le\; \sigma_{\max}(\boldsymbol{A}_{\mathsf{I}}^* \boldsymbol{A}_{\mathsf{I}}) \;\le\; 1 + \underbrace{k\mu(\boldsymbol{A})}_{\delta}. \tag{3}$$

If $\delta = k\mu(\boldsymbol{A})$ is small, all $\sigma(\boldsymbol{A}_{\mathsf{I}}^* \boldsymbol{A}_{\mathsf{I}})$ are close to 1.

Restricted Isometry Property

Definition (Restricted Isometry Property)

The matrix $\boldsymbol{A}$ satisfies the *restricted isometry property (RIP)* of *order* k, with constant $\delta \in [0,1)$, if

$$\forall\, \boldsymbol{x} \;\; k\text{-sparse}, \quad (1-\delta)\,\|\boldsymbol{x}\|_2^2 \;\le\; \|\boldsymbol{A}\boldsymbol{x}\|_2^2 \;\le\; (1+\delta)\,\|\boldsymbol{x}\|_2^2\,. \tag{4}$$

The *order-k restricted isometry constant* $\delta_k(\boldsymbol{A})$ is the smallest number δ such that the above inequality holds.

Example of Gaussian Matrices: If $\boldsymbol{A}_{\mathsf{I}}$ is a large $m \times k$ $(k < m)$ matrix with entries independent $\mathcal{N}(0, 1/m)$,

$$\sigma_{min}(\boldsymbol{A}_{\mathsf{I}}^*\boldsymbol{A}_{\mathsf{I}}) \approx (\sqrt{1} - \sqrt{k/m})^2 \ge 1 - 2\sqrt{k/m},$$

$$\sigma_{max}(\boldsymbol{A}_{\mathsf{I}}^*\boldsymbol{A}_{\mathsf{I}}) \approx (\sqrt{1} + \sqrt{k/m})^2 \le 1 + 3\sqrt{k/m}.$$

Restricted Isometry Property

Upper and lower bounds[1] for RIP constants of random Gaussian matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. Consider proportional growth of the size of (k, m, n): $\rho = \frac{k}{m}, \gamma = \frac{m}{n}$.

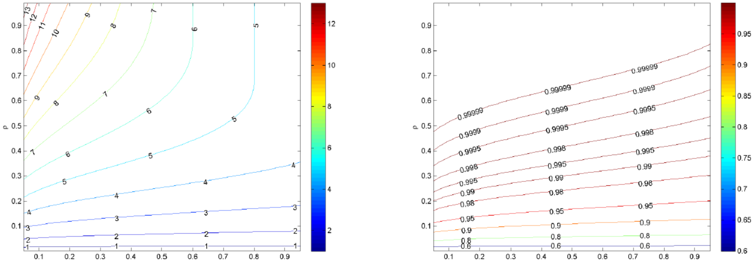

Figure: Left: $\delta_u = \lambda_{\max}(\rho, \gamma) - 1$; Right: $\delta_l = 1 - \lambda_{\min}(\rho, \gamma)$.

[1] *Improved Bounds on Restricted Isometry Constants for Gaussian Matrices,* B. Bah, J. Tanner, SIAM Journal on Matrix Analysis and Applications, 2010.

Restricted Isometry Property: Uniqueness

Theorem (ℓ^0 Recovery under RIP)

Suppose that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_o$*, with* $k = \|\boldsymbol{x}_o\|_0$*. If* $\delta_{2k}(\boldsymbol{A}) < 1$*, then* $\boldsymbol{x}_o$ *is the unique optimal solution to*

$$\min \|\boldsymbol{x}\|_0 \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}. \tag{5}$$

Proof.

Suppose on the contrary that there exists $\boldsymbol{x}' \neq \boldsymbol{x}_o$ with $\|\boldsymbol{x}'\|_0 \leq k$. Then $\boldsymbol{x}_o - \boldsymbol{x}' \in \operatorname{null}(\boldsymbol{A})$, and $\|\boldsymbol{x}_o - \boldsymbol{x}'\|_0 \leq 2k$. This implies that $\delta_{2k}(\boldsymbol{A}) \geq 1$, contradicting our assumption. □

Restricted Isometry Property: Correctness

Theorem (ℓ^1 Recovery under RIP)

Suppose that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_o$, with $k = \|\boldsymbol{x}_o\|_0$. If $\delta_{2k}(\boldsymbol{A}) < \sqrt{2} - 1$, then $\boldsymbol{x}_o$ is the unique optimal solution to

$$\min \|\boldsymbol{x}\|_1 \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}. \tag{6}$$

Some later developments:

- $\delta_{2k} < \sqrt{2} - 1 \approx 0.414$, Candes and Tao, 2006.
- $\delta_{2k} < 0.4531$, Fouchart and Lai, 2009.
- $\delta_{2k} < 0.472$, Cai, Wang, and Xu, 2009.
- $\delta_k < 0.307$, Cai, Wang, and Xu, 2010.[2]

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[2] *New Bounds for Restricted Isometry Constants*, T. Cai, L. Wang, and G. Xu, IEEE Transactions on Information Theory, 56, 2010.

Restricted Isometry Property: Universality

Computing RIP constant $\delta_k(\boldsymbol{A})$ is in general NP-hard; and in fact, certifying a matrix is (k, δ)-RIP is also hard when $k \gg \sqrt{m}$.[3] **However:**

Theorem (RIP of Gaussian Matrices)

There exists a numerical constant $C > 0$ such that if $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is a random matrix with entries independent $\mathcal{N}\left(0, \frac{1}{m}\right)$ random variables, with high probability, $\delta_k(\boldsymbol{A}) < \delta$, provided

$$m \geq Ck \log(n/k)/\delta^2. \tag{7}$$

Compare with Incoherence: With incoherence, we need $m \geq \Omega(k^2)$. Here this result allows (k, m, n) to scale proportionally:

$$m \geq \Omega(k).$$

[3] The Average-Case Time Complexity of Certifying the Restricted Isometry Property, Y. Ding, D. Kunisky, A. Wein, and A. Bandeira, https://arxiv.org/pdf/2005.11270.pdf.

ℓ^1 Recovery under RIP

How to prove ℓ^1 minimization succeeds under RIP?

Null Space Property

Given $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_o$, try to recover $\boldsymbol{x}_o$ from

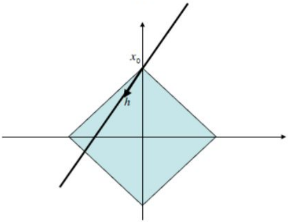
$$\min \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}. \tag{8}$$

Let $\boldsymbol{x}_{\ell^1}$ is the optimal solution. If $\boldsymbol{h} = \boldsymbol{x}_{\ell^1} - \boldsymbol{x}_o \neq \mathbf{0}$. Since $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_o = \boldsymbol{A}\boldsymbol{x}_{\ell^1}$, we also have $\boldsymbol{A}\boldsymbol{h} = \mathbf{0}$. We must have

$$\begin{aligned} 0 &\geq \|\boldsymbol{x}_{\ell^1}\|_1 - \|\boldsymbol{x}_o\|_1 = \|\boldsymbol{x}_o + \boldsymbol{h}\|_1 - \|\boldsymbol{x}_o\|_1 \\ &\geq \|\boldsymbol{x}_o\|_1 - \|\boldsymbol{h}_{\mathsf{I}}\|_1 + \|\boldsymbol{h}_{\mathsf{I}^c}\|_1 - \|\boldsymbol{x}_o\|_1 \\ &= -\|\boldsymbol{h}_{\mathsf{I}}\|_1 + \|\boldsymbol{h}_{\mathsf{I}^c}\|_1. \end{aligned}$$

That is, we have

$$\|\boldsymbol{h}_{\mathsf{I}^c}\|_1 \leq \|\boldsymbol{h}_{\mathsf{I}}\|_1.$$



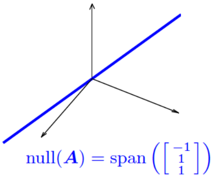
Null Space Property

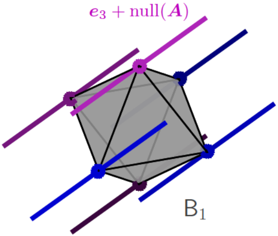
Definition (Null Space Property)

The matrix $\boldsymbol{A}$ satisfies the *null space property* of order k if for every $\boldsymbol{h} \in \mathrm{null}(\boldsymbol{A}) \setminus \{\boldsymbol{0}\}$ and every I of size at most k,

$$\|\boldsymbol{h}_{\mathsf{I}}\|_1 < \|\boldsymbol{h}_{\mathsf{I}^c}\|_1 . \tag{9}$$

Example: $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{2\times 3}$:





Null Space Property

Lemma (Success from Null Space Property)

Suppose that $\boldsymbol{A}$ satisfies the null space property of order k. Then for any $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_o$, with $\|\boldsymbol{x}_o\|_0 \le k$, $\boldsymbol{x}_o$ is the unique optimal solution to the ℓ^1 problem

$$\min \|\boldsymbol{x}\|_1 \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}. \tag{10}$$

Proof.

Let $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_o$, with $\|\boldsymbol{x}_o\|_0 \le k$, and let $\mathsf{I} = \mathrm{supp}(\boldsymbol{x}_o)$. Let $\hat{\boldsymbol{x}}_{\ell^1}$ be the optimal solution, so $\boldsymbol{h} = \hat{\boldsymbol{x}}_{\ell^1} - \boldsymbol{x}_o \in \mathrm{null}(\boldsymbol{A})$. If $\boldsymbol{h} \neq \boldsymbol{0}$, then

$$\|\hat{\boldsymbol{x}}_{\ell^1}\|_1 = \|\boldsymbol{x}_o + \boldsymbol{h}\|_1 \ge \|\boldsymbol{x}_o\|_1 - \|\boldsymbol{h}_{\mathsf{I}}\|_1 + \|\boldsymbol{h}_{\mathsf{I}^c}\|_1 > \|\boldsymbol{x}_o\|_1,$$

contradicting the optimality of $\hat{\boldsymbol{x}}_{\ell^1}$. □

No direction h in $\mathrm{null}(\boldsymbol{A})$ could further reduce $\|\boldsymbol{x}_o\|_1$.

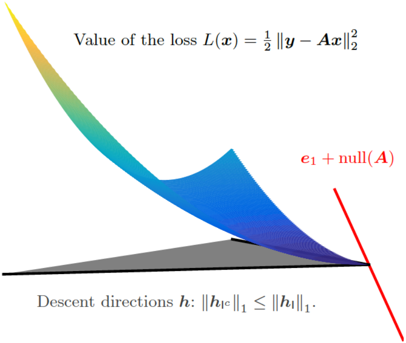
Restricted Strong Convexity Condition

The null space property is equivalent to:

$$\|\boldsymbol{A}\boldsymbol{h}\|_2^2 > 0 \quad \forall \boldsymbol{h} \;\; \|\boldsymbol{h}_{\mathsf{I}^c}\|_1 \leq \|\boldsymbol{h}_{\mathsf{I}}\|_1. \tag{11}$$

$\|\boldsymbol{A}\boldsymbol{h}\|_2^2$ must attain its minimum $\mu > 0$ on a compact set. The above is equivalent to:

$$\|\boldsymbol{A}\boldsymbol{h}\|_2^2 \geq \mu\|\boldsymbol{h}\|_2^2, \quad \forall \boldsymbol{h} \;\; \|\boldsymbol{h}_{\mathsf{I}^c}\|_1 \leq \|\boldsymbol{h}_{\mathsf{I}}\|_1. \tag{12}$$



Restricted Strong Convexity Condition

Definition (Restricted Strong Convexity)

The matrix $\boldsymbol{A}$ satisfies the *restricted strong convexity* (RSC) condition of order k, with parameters $\mu > 0$, $\alpha \geq 1$, if for every I of size at most k and for all nonzero $\boldsymbol{h}$ satisfying $\|\boldsymbol{h}_{\mathsf{I}^c}\|_1 \leq \alpha \|\boldsymbol{h}_{\mathsf{I}}\|_1$,

$$\|\boldsymbol{A}\boldsymbol{h}\|_2^2 \geq \mu \|\boldsymbol{h}\|_2^2. \tag{13}$$

Lemma (Success from RSC Condition)

Suppose that $\boldsymbol{A}$ satisfies the restricted strong convexity condition of order k with constant $\alpha \geq 1$, for some $\mu > 0$. Then for any $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_o$, with $\|\boldsymbol{x}_o\|_0 \leq k$, $\boldsymbol{x}_o$ is the unique optimal solution to the ℓ^1 problem

$$\min \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}. \tag{14}$$

RIP Preserves Incoherence

Lemma (RIP Preserves Incoherence of Images of Sparse Vectors)

If $\boldsymbol{x}$, $\boldsymbol{z}$ *are vectors with disjoint support, and* $|\mathrm{supp}(\boldsymbol{x})| + |\mathrm{supp}(\boldsymbol{z})| \le k$, *then*

$$|\langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{A}\boldsymbol{z}\rangle| \;\le\; \delta_k(\boldsymbol{A}) \left\|\boldsymbol{x}\right\|_2 \left\|\boldsymbol{z}\right\|_2 . \tag{15}$$

Proof.

WLOG, $\|\boldsymbol{x}\|_2 = \|\boldsymbol{z}\|_2 = 1$. Notice that $\|\boldsymbol{p}+\boldsymbol{q}\|_2^2 - \|\boldsymbol{p}-\boldsymbol{q}\|_2^2 = 4\langle \boldsymbol{p}, \boldsymbol{q}\rangle$. Hence,

$$|\langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{A}\boldsymbol{z}\rangle| \;\le\; \frac{1}{4}\left|\|\boldsymbol{A}\boldsymbol{x}+\boldsymbol{A}\boldsymbol{z}\|_2^2 - \|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{A}\boldsymbol{z}\|_2^2\right| \tag{16}$$

$$\le\; \frac{1}{4}\left|(1+\delta_k)\,\|\boldsymbol{x}+\boldsymbol{z}\|_2^2 - (1-\delta_k)\,\|\boldsymbol{x}-\boldsymbol{z}\|_2^2\right|. \tag{17}$$

Since $\boldsymbol{x}$ and $\boldsymbol{z}$ have disjoint support, $\|\boldsymbol{x}+\boldsymbol{z}\|_2^2 = \|\boldsymbol{x}-\boldsymbol{z}\|_2^2 = 2$. □

Bounds between Norms

Lemma (Bounds between Norms of Sparse Vectors)

For any vector z *with* $\|z\|_0 \leq k$, $\|z\|_1 \leq \sqrt{k}\,\|z\|_2$ *and* $\|z\|_2 \leq \sqrt{k}\,\|z\|_\infty$.

Proof.

First inequality: since x^2 is a convex function, we have:

$$\left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right)^2 \leq \frac{a_1^2 + a_2^2 + \cdots + a_k^2}{k}. \tag{18}$$

Second inequality:

$$\frac{a_1^2 + a_2^2 + \cdots + a_k^2}{k} \leq \max_j \{a_j^2\}. \tag{19}$$

□

RIP Implies RSC

Theorem (RIP Implies RSC)

If a matrix $\boldsymbol{A}$ satisfies RIP with $\delta_{2k}(\boldsymbol{A}) < \frac{1}{1+\alpha\sqrt{2}}$, then $\boldsymbol{A}$ satisfies the RSC condition of order k with constant α.

Proof (A Sketch): We want to show:

$$\forall \boldsymbol{h} \in \mathbb{R}^n : \|\boldsymbol{h}_{\mathsf{I}^c}\|_1 \leq \alpha \cdot \|\boldsymbol{h}_{\mathsf{I}}\|_1,\ |\mathsf{I}| = k \implies \|\boldsymbol{A}\boldsymbol{h}\|_2 \geq \mu\|\boldsymbol{h}\|_2. \tag{20}$$

Partition the indices of the entries in $\boldsymbol{h}_{\mathsf{I}^c}$ based on their magnitudes:

J_1 indexes the k largest (in magnitude) elements of $\boldsymbol{h}_{\mathsf{I}^c}$,

J_2 indexes the k largest (in magnitude) elements of $\boldsymbol{h}_{(\mathsf{I}\cup\mathsf{J}_1)^c}$,

J_3 indexes the k largest (in magnitude) elements of $\boldsymbol{h}_{(\mathsf{I}\cup\mathsf{J}_1\cup\mathsf{J}_2)^c}$,

$$\vdots$$

Then $\forall\ i \geq 1,\ \|\boldsymbol{h}_{\mathsf{J}_i}\|_1 \ \geq\ k \cdot \left\|\boldsymbol{h}_{\mathsf{J}_{i+1}}\right\|_\infty$.

RIP Implies RSC

Proof (Continued):

Step 1: Since $\boldsymbol{h}_{\mathsf{I}}$ and $\boldsymbol{h}_{\mathsf{J}_1}$ likely have the largest entries, first try to show:

$$\|\boldsymbol{A}\boldsymbol{h}\|_2 \geq C \left\|\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\right\|_2, \quad \text{for some} \quad C > 0. \tag{21}$$

From $\boldsymbol{A}\boldsymbol{h}_{\mathsf{I}} + \boldsymbol{A}\boldsymbol{h}_{\mathsf{J}_1} = \boldsymbol{A}\boldsymbol{h} - \boldsymbol{A}\boldsymbol{h}_{\mathsf{J}_2} - \boldsymbol{A}\boldsymbol{h}_{\mathsf{J}_3} - \cdots$, we have:

$$\begin{aligned}
(1-\delta_{2k})\|\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\|_2^2 \;&\leq\; \|\boldsymbol{A}\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\|_2^2 \\
&= \langle \boldsymbol{A}\boldsymbol{h}_{\mathsf{I}} + \boldsymbol{A}\boldsymbol{h}_{\mathsf{J}_1}, -\boldsymbol{A}\boldsymbol{h}_{\mathsf{J}_2} - \boldsymbol{A}\boldsymbol{h}_{\mathsf{J}_3} - \cdots \rangle + \langle \boldsymbol{A}\boldsymbol{h}_{\mathsf{I}} + \boldsymbol{A}\boldsymbol{h}_{\mathsf{J}_1}, \boldsymbol{A}\boldsymbol{h} \rangle \\
&\leq \sum_{j=2}^{\infty} \left(\left|\langle \boldsymbol{A}\boldsymbol{h}_{\mathsf{I}}, \boldsymbol{A}\boldsymbol{h}_{\mathsf{J}_j}\rangle\right| + \left|\langle \boldsymbol{A}\boldsymbol{h}_{\mathsf{J}_1}, \boldsymbol{A}\boldsymbol{h}_{\mathsf{J}_j}\rangle\right| \right) + \|\boldsymbol{A}\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\|_2 \|\boldsymbol{A}\boldsymbol{h}\|_2 \\
&\leq \delta_{2k}(\|\boldsymbol{h}_{\mathsf{I}}\|_2 + \|\boldsymbol{h}_{\mathsf{J}_1}\|_2) \sum_{j=2}^{\infty} \left\|\boldsymbol{h}_{\mathsf{J}_j}\right\|_2 + (1+\delta_{2k})^{1/2} \|\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\|_2 \|\boldsymbol{A}\boldsymbol{h}\|_2 \\
&\leq \delta_{2k}\sqrt{2} \left\|\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\right\|_2 \left\|\boldsymbol{h}_{\mathsf{I}^c}\right\|_1 / \sqrt{k} + (1+\delta_{2k})^{1/2} \|\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\|_2 \|\boldsymbol{A}\boldsymbol{h}\|_2.
\end{aligned} \tag{22}$$

RIP Implies RSC

Proof (Continued):

From the restricted cone condition, we have

$$\|\boldsymbol{h}_{\mathsf{I}^c}\|_1 \leq \alpha \|\boldsymbol{h}_{\mathsf{I}}\|_1 \leq \alpha\sqrt{k}\|\boldsymbol{h}_{\mathsf{I}}\|_2 \leq \alpha\sqrt{k}\|\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\|_2. \tag{23}$$

This gives:

$$\|\boldsymbol{A}\boldsymbol{h}\|_2 \geq \frac{1-\delta_{2k}(1+\alpha\sqrt{2})}{(1+\delta_{2k})^{1/2}}\|\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\|_2. \tag{24}$$

Step 2: Try to show:

$$\|\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\|_2^2 \geq C'\|\boldsymbol{h}\|_2^2, \quad \text{for some} \quad C' > 0. \tag{25}$$

Since the i-th element of $\boldsymbol{h}_{(\mathsf{I}\cup\mathsf{J}_1)^c}$ is no larger than the mean of the first i elements of $\boldsymbol{h}_{\mathsf{I}^c}$, we have

$$|\boldsymbol{h}_{(\mathsf{I}\cup\mathsf{J}_1)^c}|_{(i)} \leq \|\boldsymbol{h}_{\mathsf{I}^c}\|_1/i. \tag{26}$$

RIP Implies RSC

Proof (Continued):

Combining with the restriction (20), we have

$$\begin{aligned} \|\boldsymbol{h}_{(\mathsf{I}\cup\mathsf{J}_1)^c}\|_2^2 &\leq \|\boldsymbol{h}_{\mathsf{I}^c}\|_1^2 \sum_{i=k+1}^{\infty} \frac{1}{i^2} \leq \frac{\|\boldsymbol{h}_{\mathsf{I}^c}\|_1^2}{k} \\ &\leq \frac{\alpha^2\|\boldsymbol{h}_{\mathsf{I}}\|_1^2}{k} \leq \alpha^2\|\boldsymbol{h}_{\mathsf{I}}\|_2^2 \leq \alpha^2\|\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\|_2^2. \end{aligned}$$

So we have

$$\|\boldsymbol{h}\|_2^2 \leq (1+\alpha^2)\|\boldsymbol{h}_{\mathsf{I}\cup\mathsf{J}_1}\|_2^2. \tag{27}$$

Finally: Combine the results:

$$\|\boldsymbol{A}\boldsymbol{h}\|_2 \geq \frac{1-\delta_{2k}(1+\alpha\sqrt{2})}{(1+\delta_{2k})^{1/2}\sqrt{1+\alpha^2}} \|\boldsymbol{h}\|_2 \,. \tag{28}$$

RIP for ℓ^1 Minimization

To prove the RIP Theorem: We need $\delta_{2k} < \frac{1}{1+\alpha\sqrt{2}}$ to ensure RSC hence the NSP.
For ℓ^1 minimization

$$\min \|\boldsymbol{x}\|_1 \quad \text{s.t.} \quad \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}. \tag{29}$$

to succeed, we need

$$\forall \boldsymbol{h} \in \text{null}(\boldsymbol{A}) : \ \|\boldsymbol{h}_{\mathsf{I}^c}\|_1 \leq \|\boldsymbol{h}_{\mathsf{I}}\|_1,$$

hence $\alpha = 1$ and the associated RIP constant should be:

$$\delta_{2k} < \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1.$$

Conclusions

Conditions when ℓ^1 minimization find the correct k-sparse solution:

$$\min \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}.$$

- **Mutual Coherence:**

$$m = O(k^2).$$

- **Restricted Isometry:**

$$m = O\big(k \log(n/k)\big).$$

Next: what $m \times n$ matrix A has a small RIP constant δ_k?

Assignments

- Reading: Section 3.3 of Chapter 3.
- Programming Homework #1.