Computational Principles for High-dim Data Analysis (Lecture Twenty)

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Structured Nonlinear Low-Dimensional Models Transform Invariant/Equivariant Low-Rank Texture

- 1 Low-Rank Models for Images
- 2 Image Inpainting as Matrix Completion
- **3** Transform Invariant Low-Rank Textures
- 4 Applications of TILT

"What humans do with the language of mathematics is to describe patterns."

- Lynn A. Steen

Importance of Mathematical Modeling

If you formulate a problem correctly, you have more than halfway solved it!

Image: Image:

Example: Rank Conditions for Multiple-View Geometry

Geometric and algorithmic foundations for 3D Reconstruction from images.



All multi-view incidence conditions among points, lines, planes and symmetric objects are captured by the same rank condition (Ma, 2003):

$$\operatorname{rank}(\boldsymbol{M}) = 1 \text{ (or 2)}. \tag{1}$$

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Example: Rank Conditions for Photometric Stereo

Geometric and algorithmic foundations for 3D Reconstruction from images.





(b) structured lights

(c) photometric stereo

Images of a (Lambertian) object under different lighting conditions, viewed as columns of a matrix, always satisfy a rank condition (Chapter 14):

$$\mathsf{rank}(oldsymbol{M})=3.$$

Example: Low-Rank Structures in a Single Image

Low-dimensional structures arise at all spatial scales even in an individual image, especially that of man-made objects.



(a) a calibration rig



(b) a carpet



(c) windows



(d) a door



Be aware though: only at a rectified view and without any occlusion!

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Mathematical Model for Low-Rank Image (Regions)

An image (region) $I_o(x, y)$ can be explicitly factorized as as the combination of r rank-1 functions:

$$\boldsymbol{I}_{o}(x,y) \doteq \sum_{i=1}^{r} u_{i}(x) \cdot v_{i}(y).$$
(3)

Symmetry: two low-rank textures *equivalent* if they are scaled and translated versions of each other:

$$I_o(x,y) \sim \alpha \cdot I_o(ax+t_1, by+t_2),$$

for some $\alpha, a, b, c \in \mathbb{R}_+, t_1, t_2 \in \mathbb{R}$.

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Image Inpainting as Matrix Completion

Example: Low-rank image inpainting as matrix completion:

$$\min_{\boldsymbol{L}} \operatorname{rank}(\boldsymbol{L}) \quad \text{s.t.} \quad \boldsymbol{L}(i,j) = \boldsymbol{I}_o(i,j) \ \forall (i,j) \in \Omega.$$
(4)



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Image Inpainting as Matrix Completion



Is low-rank the only low-dim structure in such images?

$$\boldsymbol{I}_o(x,y) = \sum_{i=1}^r u_i(x)v_i(y) = \boldsymbol{U}\boldsymbol{V}^*.$$

 $m{U}$ and $m{V}$ might themselves be sparse in some bases $m{U}=m{B}_1m{X}_1, m{V}=m{B}_2m{X}_2$:

 $I_o = B_1 X_1 X_2^* B_2^* \doteq B_1 W_o B_2^*$, with $W_o = X_1 X_2^*$ sparse.

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Image Inpainting as Matrix Completion

Impose low-rank and sparse structures simultaneously (not RPCA!):

$$\min_{\boldsymbol{L},\boldsymbol{W}} \operatorname{rank}(\boldsymbol{L}) + \lambda \|\boldsymbol{W}\|_{0} \quad \text{s.t.} \quad \mathcal{P}_{\Omega}[\boldsymbol{L}] = \mathcal{P}_{\Omega}[\boldsymbol{I}], \ \boldsymbol{L} = \boldsymbol{B}_{1}\boldsymbol{W}\boldsymbol{B}_{2}^{*}.$$
(5)

Relaxation to a corresponding convex program (why?):

$$\min_{\boldsymbol{W}} \|\boldsymbol{W}\|_* + \lambda \|\boldsymbol{W}\|_1 \quad \text{s.t.} \quad \mathcal{P}_{\Omega}[\boldsymbol{B}_1 \boldsymbol{W} \boldsymbol{B}_2^*] = \mathcal{P}_{\Omega}[\boldsymbol{I}]. \tag{6}$$

Reformulation for a better optimization algorithm:

 $\min_{\boldsymbol{L},\boldsymbol{W}} \|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{W}\|_1 \quad \text{s.t.} \quad \boldsymbol{L} = \boldsymbol{W}, \ \mathcal{P}_{\Omega}[\boldsymbol{B}_1 \boldsymbol{W} \boldsymbol{B}_2^*] = \mathcal{P}_{\Omega}[\boldsymbol{I}].$ (7)

Note that the above problem can be solved by ALM + ADMM!

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Image Inpainting: Improved Performance



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Image Recovery

Robustness to corruption $I = I_o + E_o$ for unknown sparse E_o :

 $\min_{\boldsymbol{W}} \|\boldsymbol{W}\|_* + \lambda \|\boldsymbol{W}\|_1 + \alpha \|\boldsymbol{E}\|_1 \quad \text{s.t.} \quad \mathcal{P}_{\Omega}[\boldsymbol{B}_1 \boldsymbol{W} \boldsymbol{B}_2^* + \boldsymbol{E}] = \mathcal{P}_{\Omega}[\boldsymbol{I}].$ (8)



Figure: Left: Low-rank; Middle: Microsoft; Right: Adobe.

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Deformation and Corruption

In practice, the observe image I is typically a transformed version of the (rectified) low-rank image I_o : $I \circ \tau(x, y) = I_o(x, y)$ or:

$$\boldsymbol{I}(x,y) = \boldsymbol{I}_o \circ \tau^{-1}(x,y) = \boldsymbol{I}_o \left(\tau^{-1}(x,y) \right), \quad \tau \in \mathbb{G}.$$



(a) Low-rank texture \boldsymbol{I}_o



(b) Its image \boldsymbol{I} under a different viewpoint

Note: the observed image might also be corrupted: $I = (I_o + E_o) \circ \tau^{-1}$.

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Algorithm Development

Original problem: transformed low-rank and sparse decomposition

 $\min_{\boldsymbol{L},\boldsymbol{E},\tau} \operatorname{rank}(\boldsymbol{L}) + \gamma \|\boldsymbol{E}\|_0 \quad \text{subject to} \quad \boldsymbol{I} \circ \tau = \boldsymbol{L} + \boldsymbol{E}. \tag{9}$

Relaxation: use convex surrogates for rank and sparsity

$$\min_{\boldsymbol{L},\boldsymbol{E},\tau} \underbrace{\|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{E}\|_1}_{\text{convex}} \quad \text{subject to} \quad \underbrace{\boldsymbol{I} \circ \tau = \boldsymbol{L} + \boldsymbol{E}}_{\text{still nonlinear!}}.$$
(10)

Linearization to deal with nonlinearity: $\pmb{I} \circ \tau + \nabla \pmb{I} \cdot d\tau \approx \pmb{L} + \pmb{E}$ and

 $\min_{\boldsymbol{L},\boldsymbol{E},d\tau} \|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{E}\|_1 \text{ subject to } \boldsymbol{I} \circ \tau + \nabla \boldsymbol{I} \cdot d\tau = \boldsymbol{L} + \boldsymbol{E}.$ (11)

Note: this is exactly a **Compressive PCP** problem (in Chapter 5)!

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The TILT Algorithm

INPUT: Input image $I \in \mathbb{R}^{w \times h}$, initial transformation $\tau \in \mathbb{G}$ (affine or projective), and a weight $\lambda > 0$. WHILE not converged **DO**

Step 1: Normalization and compute Jacobian:

$$\boldsymbol{I} \circ \tau \leftarrow \frac{\boldsymbol{I} \circ \tau}{\|\boldsymbol{I} \circ \tau\|_{F}}; \quad \nabla \boldsymbol{I} \leftarrow \frac{\partial}{\partial \zeta} \left(\frac{\operatorname{vec}(\boldsymbol{I} \circ \zeta)}{\|\operatorname{vec}(\boldsymbol{I} \circ \zeta)\|_{2}} \right) \Big|_{\zeta = \tau}; \qquad (12)$$

Step 2 (inner loop): Solve the linearized CPCP problem:

$$\begin{array}{rcl} (\boldsymbol{L}_{\star}, \boldsymbol{E}_{\star}, d\tau_{\star}) & \leftarrow & \arg\min_{\boldsymbol{L}, \boldsymbol{E}, d\tau} & \|\boldsymbol{L}\|_{*} + \lambda \|\boldsymbol{E}\|_{1} \\ & & \text{subject to} & \boldsymbol{I} \circ \tau + \nabla \boldsymbol{I} \cdot d\tau = \boldsymbol{L} + \boldsymbol{E}; \end{array}$$
(13)

Step 3: Update the transformation: $\tau \leftarrow \tau + d\tau_{\star}$; **END WHILE OUTPUT:** Converged solution L_{\star} , E_{\star} , τ_{\star} to problem (10).

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Inner Loop of TILT

Apply ALM + ADMM to solve the inner loop CPCP problem (13):

INPUT: The current (deformed and normalized) image $I \circ \tau \in \mathbb{R}^{m \times n}$ and its Jacobian ∇I against current deformation τ (from the outer loop), and $\lambda > 0$.

Initialization: $k = 0, Y_0 = 0, E_0 = 0, d\tau_0 = 0, \mu_0 > 0, \rho > 1$; WHILE not converged **DO**

$$\begin{aligned} (\boldsymbol{U}_{k},\boldsymbol{\Sigma}_{k},\boldsymbol{V}_{k}) &= \mathrm{SVD}\big(\boldsymbol{I}\circ\tau+\nabla\boldsymbol{I}\cdot d\tau_{k}-\boldsymbol{E}_{k}+\mu_{k}^{-1}\boldsymbol{Y}_{k}\big);\\ \boldsymbol{L}_{k+1} &= \boldsymbol{U}_{k}\mathrm{soft}(\boldsymbol{\Sigma}_{k},\mu_{k}^{-1})\boldsymbol{V}_{k}^{*};\\ \boldsymbol{E}_{k+1} &= \mathrm{soft}(\boldsymbol{I}\circ\tau+\nabla\boldsymbol{I}\cdot d\tau_{k}-\boldsymbol{L}_{k+1}+\mu_{k}^{-1}\boldsymbol{Y}_{k},\lambda\mu_{k}^{-1});\\ d\tau_{k+1} &= (\nabla\boldsymbol{I})^{\dagger}(-\boldsymbol{I}\circ\tau+\boldsymbol{L}_{k+1}+\boldsymbol{E}_{k+1}-\mu_{k}^{-1}\boldsymbol{Y}_{k});\\ \boldsymbol{Y}_{k+1} &= \boldsymbol{Y}_{k}+\mu_{k}(\boldsymbol{I}\circ\tau+\nabla\boldsymbol{I}\cdot d\tau_{k+1}-\boldsymbol{L}_{k+1}-\boldsymbol{E}_{k+1});\\ \mu_{k+1} &= \rho\mu_{k};\\ \textbf{END WHILE}\\ \textbf{OUTPUT: Converged solution} \ (\boldsymbol{L}_{\star}, \ \boldsymbol{E}_{\star}, \ d\tau_{\star}) \text{ to problem (11)}. \end{aligned}$$

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Rectifying Planar Low-Rank Textures



(a) Low-rank texture \boldsymbol{I}_o



(b) Its image \boldsymbol{I} under a different viewpoint

 I_o is on a planar surface and I is seen from a different view:

$$\boldsymbol{I}(x,y) = \boldsymbol{I}_o \circ \tau^{-1}(x,y) = \boldsymbol{I}_o \left(\tau^{-1}(x,y) \right)$$

where τ is any planar homography $H \in GL(3)$:

$$\tau(x,y) = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$
 (14)

Empirical Observations: large range of attraction























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Rectifying Generalized Cylinder Surfaces

In practice, very often the low-rank texture I_o does not lie on a planar surface, instead say on a cylindrical surface:



The overall deformation consists of composition of three mappings:

- from a flat 2D plane to the 3D cylindrical surface (shape);
- from the surface to the image plane (camera pose and projection);
- from the image plane to the pixel coordinates (intrinsic calibration).

Rectifying Generalized Cylinder Surfaces

Then $I_o = I \circ g$ where the transformation g from the texture coordinates $I_o(x, y)$ to the image coordinates I(u, v) is a composition of the three mappings specified above:

$$g: (x, y) \mapsto (X_c, Y_c, Z_c) \mapsto (x_n, y_n) \mapsto (u, v).$$
(15)

Hence we solve the following nonlinear optimization problem:

$$\min_{\boldsymbol{L},\boldsymbol{E},\boldsymbol{c},\boldsymbol{R},\boldsymbol{T}} \|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{E}\|_1 \quad \text{s.t.} \quad \boldsymbol{I} \circ \boldsymbol{g} = \boldsymbol{L} + \boldsymbol{E}.$$
(16)

As in TILT, this can be solved iteratively from its linearized version:

$$\min_{\boldsymbol{L},\boldsymbol{E},dg} \|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{E}\|_1 \quad \text{s.t.} \quad \boldsymbol{I} \circ \boldsymbol{g} + \nabla \boldsymbol{I}_{\boldsymbol{g}} \cdot d\boldsymbol{g} = \boldsymbol{L} + \boldsymbol{E},$$
(17)

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Empirical Results















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Camera Calibration

Three most important factors in robotics and augmented reality: calibration, calibration, calibration!



Figure 15.10 Left: typical image of a fisheye camera. Right: image of a perspective camera.



Figure 15.11 Left two: Images of a typical calibration rig. Right: Corners need to be marked (or detected) for conventional calibration methods.

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Camera Calibration: from a calibration rig to the image

Extrinsic parameters τ_i (camera pose – different for each image *i*):

rotation $\mathbf{R} \in SO(3)$ and translation $\mathbf{T} \in \mathbb{R}^3$.

Intrinsic parameters τ_0 (focal length etc. – common for all images):

$$\boldsymbol{K} \doteq \begin{bmatrix} f_x & \theta & o_x \\ 0 & f_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 3}.$$
 (18)

Lens distortion τ_0 (radial distortion etc. – common for all images):

$$\begin{cases} r \doteq \sqrt{x_n^2 + y_n^2}, \\ f(r) \doteq 1 + k_c(1)r^2 + k_c(2)r^4 + k_c(5)r^6, \\ p_d = \begin{bmatrix} f(r)x_n + 2k_c(3)x_ny_n + k_c(4)(r^2 + 2x_n^2) \\ f(r)x_n + 2k_c(4)x_ny_n + k_c(3)(r^2 + 2y_n^2) \end{bmatrix}. \end{cases}$$
(19)

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Camera Calibration



Multiple images I_i of the calibration rig I_o at different views τ_i are related by:

$$\boldsymbol{I}_i \circ (\tau_0 \circ \tau_i) = \boldsymbol{I}_o + \boldsymbol{E}_i, \quad i = 1, 2, \dots, N.$$
(20)

min
$$\sum_{i=1}^{N} \|L_i\|_* + \|E_i\|_1$$
, s.t. $I_i \circ (\tau_0 \circ \tau_i) = L_i + E_i$, $L_i = L_j$. (21)

Or min $\|L_c\|_* + \|L_r\|_* + \lambda \|E\|_1$, s.t. $I_i \circ (\tau_0 \circ \tau_i) = L_i + E_i$. (22)

Calibration Results



(a) Input image with an initial window

(b) Lens distortion removed



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Summary

A good idea deserves a good implementation.

The better the idea, the better should be the implementation.

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Assignments

- Reading: Chapter 15. •
- Programming Homework #4.

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