Consider square matrices with dim $n \times n$.

Observe matrix M

PCP

$$\min \quad \|L\|_* + \lambda \|S\|_1 \qquad \lambda = \frac{1}{\sqrt{n}}$$

$$\text{s.t.} \quad L + S = M$$

Assumptions

$$L_0 = U \Sigma V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

(3) $\|UV^*\|_\infty \le \sqrt{\frac{\mu r}{n^2}}$

(1) $\max_i \|U^* e_i\|^2 \le \frac{\mu r}{n}$ (2) $\max_i \|V^* e_i\|_2^2 \le \frac{\mu r}{n}$

$P_U \doteq UU^*$

$\max_i \|P_U e_i\|^2 \le \frac{\mu r}{n}$

Thm: Support of S_0 is uniformly distributed among all sets with cardinality m. Then

PCP solves $\hat{L} = L_0$

$\hat{S} = S_0$

if $\mathrm{rank}(L_0) \leq \rho_r \frac{n}{\mu (\log n)^2}$

$\boxed{m \leq \rho_s n^2}$

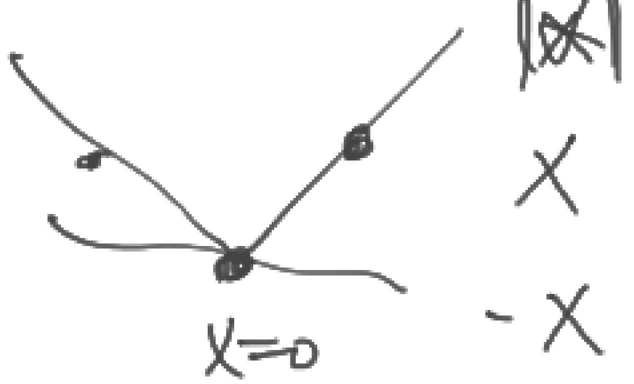
$$M_{ij} = \begin{cases} (L_0)_{ij} & \text{w.p. } 1-\rho_s \\ (L_0)_{ij} + (S_0)_{ij} & \text{w.p. } \rho_s \end{cases}$$

Bernoulli model
each entry corrupted with prob ρ_s ind.

$\boxed{\text{Proof}}$: $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$ $\quad$ $\operatorname{sign}(S)$ entry wise

Subgradient of $\|S_0\|_1$ is $\operatorname{sgn}(S_0) + F$

where $P_\Omega F = 0$ $\quad$ $\|F\|_\infty \leq 1$ $\quad$ Ω: support of S_0



$[-1,1]$

Subgradient of $\|L_0\|_*$

$L_0 = U\Sigma V^*$ compact SVD $\quad$ $U, V \in \mathbb{R}^{n\times r}$

Subgradient at L_0 is

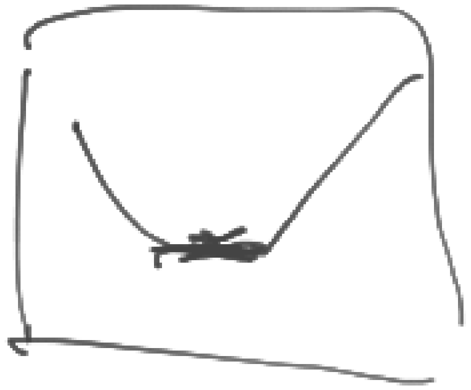
$$UV^* + W \text{ where } \boxed{P_T W = 0} \quad \boxed{\|W\| \leq 1}$$

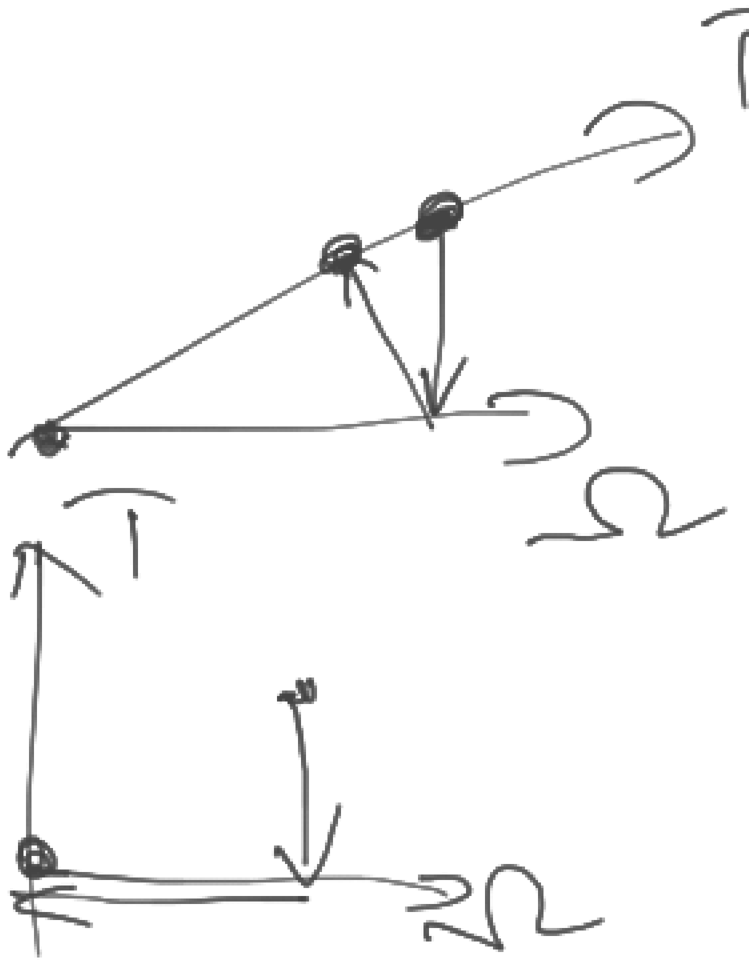
$$T = \{UX^* + YV^* : X, Y \in \mathbb{R}^{n\times r}\}$$

Lemma: Assume $\|\mathcal{P}_\Omega \mathcal{P}_T\| < 1$ $\Longleftrightarrow$ $\boxed{\Omega \cap T = \{0\}}$

$\mathcal{P}_T X = \underset{g}{\operatorname{argmin}} \{ \|X - g\|_F : g \in T \}$

$\|A\| = \sup_{X \neq 0} \frac{\|AX\|_F}{\|X\|_F}$





(L_0, S_0) is the unique solution to PCP

if there exist (W, F) s.t.

$$\boxed{UV^* + W = \lambda(\operatorname{sgn}(S_0) + F)}$$

$\mathcal{P}_\Omega F = 0 \quad \|F\|_\infty < 1$

$\mathcal{P}_T W = 0 \quad \|W\| < 1$

Proof of lemma:

Consider perturbation (L_0+H, S_0-H) want to show

$H \neq 0 \Rightarrow$ obj. strictly increases

Let $UV^* + W_0$ be some subgradient of $\|L_0\|_*$

let $\mathrm{sgn}(S_0) + F_0$ be some subgradient of $\|S_0\|_1$

Def of subgradient:

$$\|L_0+H\|_* + \lambda\|S_0-H\|_1 \geq \|L_0\|_* + \lambda\|S_0\|_1 + \langle UV^* + W_0, H\rangle - \lambda\langle \mathrm{sgn}(S_0) + F_0, H\rangle \quad (*)$$

Choose W_0 s.t. $\langle W_0, H \rangle = \|P_{T^\perp} H\|_*$

F_0 s.t. $\langle F_0, H \rangle = -\|P_{\Omega^\perp} H\|_1$

$F_0 = -\mathrm{sgn}(P_{\Omega^\perp} H)$ $\quad P_\Omega F_0 = 0 \quad \|F_0\|_\infty \le 1$ ✓

$\|P_{T^\perp} H\|_* = \langle W, P_{T^\perp} H \rangle \quad \|W\| \le 1$

$P_T W = 0 \quad W_0 = \boxed{P_{T^\perp}(W)}$

$+\langle UV^* - \lambda\,\mathrm{sgn}(S_0), H \rangle$

$\|L_0 + H\|_* + \lambda\|S_0 - H\|_1 \ge \|L_0\|_* + \lambda\|S_0\|_1 + \|P_{T^\perp} H\|_* + \lambda\|P_{\Omega^\perp} H\|_1$

$$\|P_{T^\perp} H\|_* + \lambda \|P_{\Omega^\perp} H\|_1 + \underline{\langle UV^* - \lambda \,\mathrm{sgn}(S_0), H\rangle}$$

exist $\qquad\qquad\qquad\qquad\qquad\qquad > 0$

(W, F) s.t. $\underline{UV^* + W = \lambda(\mathrm{sgn}(S_0) + F)}$ $\qquad$ if $H \neq 0$

$P_T W = 0 \quad P_\Omega F = 0 \quad \|W\| < 1 \quad \|F\|_\infty < 1$

$$UV^* - \lambda \,\mathrm{sgn}(S_0) = \lambda F - W \qquad \boxed{\beta = \max(\|W\|, \|F\|_\infty) < 1}$$

$$|\langle UV^* - \lambda \,\mathrm{sgn}(S_0), H\rangle| = |\langle \lambda F - W, H\rangle| \leq |\underline{\langle W, H\rangle}| + |\underline{\langle F, H\rangle}|$$

$$\leq \underline{\|W\| \, \|P_{T^\perp} H\|_*} + \lambda \, \underline{\|F\|_\infty \, \|P_{\Omega^\perp} H\|_1} \leq \beta(\|P_{T^\perp} H\|_* + \lambda \|P_{\Omega^\perp} H\|_1$$

$$\geq (1-\beta)\left(\|P_{T^\perp} H\|_* + \lambda \|P_{\Omega^\perp} H\|_1\right)$$

$\beta < 1$ $\qquad$ $\exists? \neq 0$ $\qquad$ > 0

$H \neq 0 \Rightarrow \boxed{H \notin T, \quad H \notin \Omega}$

$\boxed{T \cap \Omega = \{0\}} \Rightarrow \{ H \neq 0 \Rightarrow$ strict increase of obj $\}$

Suffices to find W s.t.

$$\begin{cases} \underline{W \in T^\perp} \\ \|W\| < 1 \\ P_\Omega(UV^* + W) = \lambda \operatorname{sgn}(S_0) \\ \|P_{\Omega^\perp}(UV^* + W)\|_\infty < \lambda \end{cases}$$

Lemma (David Gross)

$\|P_\Omega P_T\| \leq \frac{1}{2}$ $\lambda < 1$, then (L_0, S_0) is the unique solution if there exist a pair (W, F)

s.t.

$$UV^* + W = \lambda(\mathrm{sgn}(S_0) + F + P_\Omega D)$$

$\|P_\Omega D\|_F$

s.t. $P_T W = 0$, $\|W\| \leq \frac{1}{2}$, $P_\Omega F = 0$, $\|F\|_\infty \leq \frac{1}{2}$, $\leq \frac{1}{4}$

Suffices to find a W s.t.

$$\begin{cases} W \in T^{\perp} \\ \|W\| < \frac{1}{2} \\ \|P_{\Omega}(UV^* - \lambda \operatorname{sgn}(S_0) + W)\|_F \le \frac{\lambda}{4} \\ \|P_{\Omega^{\perp}}(UV^* + W)\|_{\infty} < \frac{\lambda}{2} \end{cases}$$

$\boxed{\text{Golfing scheme.}}$

$\Omega \sim \mathrm{Bern}(p) \qquad \Omega^c \sim \mathrm{Bern}(1-p)$

$\boxed{\Omega^c = \boxed{\Omega_1 \cup \Omega_2 \cup \Omega_3 \cdots \cup \Omega_{j_0}}} \quad \Omega_j \sim \mathrm{Bern}(q)$

ind.

p = prob of corruption $= \mathbb{P}\{B(j_0, q) = 0\} = (1-q)^{j_0}$

$p = (1-q)^{j_0}$

$$W = W^L + W^S$$

W^L: golfing scheme: $Y_0 = 0$

$$Y_j = \underline{Y_{j-1} + q^{-1} P_{\Omega_j} P_T (uv^* - Y_{j-1})}$$

$1 \le j \le j_0$

$$W^L = P_{T^\perp} Y_{j_0}$$

Construct W^S $\qquad \|P_\Omega P_T\| < \frac{1}{2}$

$\Rightarrow \|P_\Omega P_T P_T P_\Omega\| < \frac{1}{4}$ $\qquad \boxed{\|AB\| \le \|A\| \|B\|}$

$\boxed{P_\Omega} - \boxed{P_\Omega P_T P_\Omega}$ is $\boxed{\text{invertible map on } \Omega}$

$$W^S = \lambda P_{T^\perp} (P_\Omega - P_\Omega P_T P_\Omega)^{-1} \boxed{\mathrm{sgn}(S_0)}$$

Supported on Ω

$$P_\Omega(W^S) = \lambda P_\Omega \underbrace{(I - P_T)}_{P_T^{\perp}} (P_\Omega - P_\Omega P_T P_\Omega)^{-1} \mathrm{sgn}(S_0)$$

$$\lambda (P_\Omega - P_\Omega P_T P_\Omega)(P_\Omega - P_\Omega P_T P_\Omega)^{-1} \mathrm{sgn}(S_0)$$

$$= \lambda\, \mathrm{sgn}(S_0)$$

$$W = W^L + W^S$$

Matrix Completion with Grossly corrupted data

$$Y = P_{\Omega_{obs}}(L_0 + S_0)$$

$$\min \quad \|L\|_* + \lambda \|S\|_1$$

$$\text{s.t.} \quad P_{\Omega_{obs}}(L+S) = Y$$

Thm: $n \times n$ matrix, Ω_{obs} unif. dist with card. $m = \frac{n^2}{10}$, S_0 entries Bern(τ)

Then w.p. $1 - c n^{-10}$, PCP works with

$$\lambda = \frac{1}{\sqrt{n/10}}$$ from $m = 0.1 n^2$ samples

as long as $\mathrm{rank}(L_0) \le \rho_r \frac{n}{\mu^{-1}(\log n)^2}$, $\tau \le \tau_s$